

THE AUTOMORPHISM GROUP OF AN ABELIAN p -GROUP AND ITS NORMAL p -SUBGROUPS

BY

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ABSTRACT. Let Γ be the automorphism group of a nonelementary reduced abelian p -group, $p \geq 5$. It is shown that every noncentral normal subgroup of Γ contains a noncentral normal subgroup Δ of Γ such that $\Delta^p = 1$. Furthermore, every cyclic normal subgroup of Γ is contained in the center of Γ .

1. **The results.** Throughout this article, G denotes a reduced abelian p -group for some prime $p \geq 5$ and AG its automorphism group. It is well known that 1 is the only normal p -subgroup of AG if G is elementary abelian (see [6, p. 409]). However, if $pG \neq 0$ and G is not cyclic, then AG does in fact possess a large amount of noncentral (i.e. not contained in the center of AG) normal p -subgroups. This follows from our investigations in [7] where we determined the class of all (not necessarily reduced) abelian p -groups G such that every noncentral normal subgroup of AG contains a noncentral normal p -subgroup of AG .

The purpose of this article is to improve earlier results (obtained in [7]) for the special case of reduced groups. We shall prove the following theorem.

Theorem A. *For a noncyclic reduced abelian p -group G , where $p \geq 5$, the following conditions are equivalent.*

- (i) $pG \neq 0$.
- (ii) AG contains a normal p -subgroup $\neq 1$.
- (iii) AG contains noncentral normal p -subgroups.
- (iv) Every noncentral normal subgroup of AG contains a noncentral normal subgroup of AG of exponent p .

The exponent of a group X is defined to be the least positive integer n such that $x^n = 1$ for all $x \in X$.

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The proof of Theorem A will be easily completed once we have established the following result which is quite interesting by itself.

Theorem B. *If G is a reduced abelian p -group, $p \geq 5$, then every cyclic normal subgroup of AG is contained in the center of AG .*

2. Preliminaries. Our notation and terminology concerning abelian groups will be standard (see e.g. [3]). Mappings will be written to the right. If there is no danger of confusion we may not distinguish between different identity mappings and write 1 instead of 1_G . The following symbols will be used.

$$G[p] = \{g \in G \mid pg = 0\} \text{ [socle of } G],$$

$$o(x) = \text{order of } x,$$

$$\langle x \rangle = \text{cyclic subgroup generated by } x,$$

$$H \oplus K = \text{direct sum of } H \text{ and } K,$$

$$AG = \text{automorphism group of } G,$$

$$zAG = \text{center of } AG,$$

$$c\Delta = \{\alpha \in AG \mid \alpha\delta = \delta\alpha \text{ for all } \delta \in \Delta\} \text{ [centralizer of } \Delta \leq AG \text{ in } AG],$$

$$\alpha|_S = \text{restriction of } \alpha \in AG \text{ to } S \leq G,$$

$$R_p^* = \text{group of units in the ring of } p\text{-adic integers.}$$

A subgroup S of a group X is called *noncentral* if S is not contained in the center of X . For n an integer, X^n denotes the subgroup of X generated by all x^n where $x \in X$. The exponent of X is the least positive integer n such that $X^n = 1$.

Throughout this article, G will denote a reduced abelian p -group, $p \geq 5$.

In the course of our proofs, a number of well-known facts on the automorphism group of G will be used constantly. They are collected here for the convenience of the reader.

(2.1) *The center of AG .* The center of AG consists precisely of the multiplication with p -adic units, i.e.

$$zAG = R_p^* \cdot 1_G$$

[1, pp. 110, 111]. If G is unbounded then $zAG \cong R_p^*$, and the center of AG contains no element of order p . If G is bounded of exponent p^{m+1} , where $m \geq 0$, then zAG is a cyclic group of order $p^m(p-1)$. An automorphism α of G belongs to zAG if and only if $x\alpha \in \langle x \rangle$ for all cyclic direct summands $\langle x \rangle$ of G [5, p. 201]. Furthermore, AG is commutative if and only if G is cyclic (cf. [3, p. 222]).

(2.2) *The normal p -subgroups of AG .* Let

$$F^n G = (p^n G)[p]/(p^{n+1} G)[p]$$

denote the n th Ulm-factor of G . The maximal normal p -subgroup $P_{\max}$ of AG consists of all torsion elements $\alpha \in AG$ such that α induces the 1-automorphism in $F^n G$ for all integers $n \geq 0$ [6, p. 412]. If α induces the identity mapping in *all* Ulm-factors of G , then the conditions $\alpha^{p^n} = 1$ and $\alpha|p^n G = 1$ are equivalent [10, p. 101]. In particular, if $p^n G = 0$, then every automorphism of G fixing $G[p]$ elementwise belongs to $P_{\max}$ and, therefore, has order a power of p .

3. The proofs. We start out with a proof of Theorem B for a special case. In the following lemma no restrictions are imposed upon p .

Lemma 3.1. *Every normal subgroup of AG of order p is contained in the center of AG .*

Proof. Let N be a normal subgroup of AG of order p . Then $N = \langle \eta \rangle$, $o(\eta) = p$, is cyclic, and the automorphism group AN of N has order $p - 1$. Since AG/cN is isomorphic to a group of automorphisms of N , it follows that

$$(3.1) \quad \eta \cdot \alpha^{p-1} = \alpha^{p-1} \cdot \eta \quad \text{for all } \alpha \in AG,$$

and

$$(3.2) \quad \eta\alpha = \alpha\eta \quad \text{for all } \alpha \in AG \text{ of } p\text{-power order.}$$

Assume, by way of contradiction, that

$$(3.3) \quad \eta \notin zAG.$$

In [5] we have shown that (3.1) is valid for $\eta \notin zAG$ only if G is bounded (p. 206) and, also, that $\eta \notin zAG$ implies $x\eta \notin \langle x \rangle$ for some cyclic direct summand $\langle x \rangle$ of G (cf. (2.1)). Hence (3.3) implies the existence of a decomposition $G = \langle x \rangle \oplus \langle y \rangle \oplus C$ such that

$$(3.4) \quad x\eta = kx + ly + c, \quad c \in C,$$

for some integers k and l , and

$$(3.5) \quad ly \neq 0.$$

We construct an endomorphism σ of G as follows. If $o(x) > o(y)$, choose any $x' \in \langle x \rangle$ such that $o(x') = o(y)$; if $o(x) \leq o(y)$, let $x' = x$. In either case, G has an endomorphism σ such that

$$(3.6) \quad x\sigma = 0, \quad y\sigma = x', \quad C\sigma = 0.$$

Since $\sigma^2 = 0$, it follows that $\alpha = 1_G + \sigma$ is an automorphism of G the order of which is equal to the (additive) order of σ and therefore a power of p (cf. [3, p. 221]). From (3.2) we obtain

$$(3.7) \quad \eta\alpha = \alpha\eta.$$

If $o(x) > o(y)$, then $o(y) = o(x')$ by construction, and $lx' \neq 0$ because of (3.5). This together with (3.4) and (3.6) implies

$$x\eta\alpha = x\eta + lx' \neq x\eta = x\alpha\eta,$$

contradicting (3.7). Suppose that $o(x) \leq o(y)$. Then $x' = x$ and, because of (3.6) and (3.7),

$$y\eta + x\eta = y\alpha\eta = y\eta\alpha = y\eta(1_G + \sigma) = y\eta + y\eta\sigma.$$

Consequently,

$$x\eta = y\eta\sigma \in G\sigma = \langle x' \rangle = \langle x \rangle,$$

contrary to (3.4) and (3.5). Thus assumption (3.3) has led to a contradiction, proving the lemma.

We are now in a position to give a proof of Theorem B.

Proof of Theorem B. Clearly, the proposition holds true if AG is abelian. Therefore we can assume that G is not cyclic. If $pG = 0$, then G is a vector space over the prime field of characteristic p and, since $p \neq 2, 3$, every non-central normal subgroup of AG contains the group of all linear transformations of G of determinant 1 (see [2, pp. 41, 45]), which is not cyclic. Therefore, we can restrict ourselves to the case $pG \neq 0$. Let $\langle \xi \rangle$ be a cyclic normal subgroup of AG and assume, by way of contradiction, that

$$(3.8) \quad \langle \xi \rangle \not\leq \mathbf{z}AG.$$

According to [7], every noncentral normal subgroup of AG contains a noncentral normal p -subgroup of AG . Therefore, without loss of generality, we can assume that

$$(3.9) \quad \langle \xi \rangle \text{ is a } p\text{-group.}$$

Let $\eta \in \langle \xi \rangle$ be an element of order p . Since $\langle \eta \rangle$ is a characteristic subgroup of $\langle \xi \rangle$ and $\langle \xi \rangle$ is normal in AG , it follows that $\langle \eta \rangle$ is a normal subgroup of AG of order p . From Lemma 3.1 we obtain $\eta \in \mathbf{z}AG$, and consequently

$$(3.10) \quad \alpha^{-1}\eta\alpha = \eta \quad \text{for all } \alpha \in AG.$$

Clearly, $AG/\mathbf{c}\langle \xi \rangle$ is isomorphic to a group Φ of automorphisms of $\langle \xi \rangle$. It follows from (3.10) that every $\phi \in \Phi$ fixes the socle $\langle \eta \rangle$ of $\langle \xi \rangle$ elementwise and, therefore, the order of ϕ is a power of p (cf. (2.2)). Hence $AG/\mathbf{c}\langle \xi \rangle \simeq \Phi \leq A\langle \xi \rangle$ is a finite p -group. In particular, $AG/\mathbf{c}\langle \xi \rangle$ is finite and

$$(3.11) \quad (p-1) \nmid [AG : \mathbf{c}\langle \xi \rangle].$$

In [5] we have shown that for every noncentral normal subgroup Γ of AG such that $AG/c\Gamma$ is finite, $p-1$ divides the index $[AG:c\Gamma]$ of the centralizer $c\Gamma$ of Γ in AG (p. 214). Therefore, (3.11) implies $\langle \xi \rangle \leq zAG$, which is the desired contradiction to (3.8). This completes the proof of Theorem B.

Theorem B enables us to establish the following result which is the essential part of Theorem A.

Theorem 3.2. *Let G be a reduced abelian p -group for some prime $p \geq 5$ and let $pG \neq 0$. Then every noncentral normal subgroup of AG contains a noncentral normal subgroup of AG of exponent p .*

Proof. Let N be a noncentral normal subgroup of AG . Then N contains a normal p -subgroup N_1 of AG such that

$$(3.12) \quad N_1 \not\leq zAG$$

(see [7]). For elements γ in the maximal normal p -subgroup of AG , the conditions $\gamma^p = 1$ and $\gamma|pG = 1$ are known to be equivalent (cf. (2.2)). Hence

$$(3.13) \quad P = \{\gamma \in N_1 \mid \gamma^p = 1\} = N_1 \cap \{\alpha \in AG \mid (\alpha|pG) = 1\}$$

is a normal subgroup of AG of exponent p which is contained in N . Assume, by way of contradiction, that

$$(3.14) \quad P = \{\gamma \in N_1 \mid \gamma^p = 1\} \text{ is cyclic.}$$

Then $P \leq zAG$ according to Lemma 3.1 and, since $P \neq 1$, it follows that $zAG = R_p^* \cdot 1_G$ has elements of order p . Hence $p^n G = 0$ (see (2.1)) and therefore

$$(3.15) \quad N_1^{p^n} = 1$$

for some positive integer n (cf. (2.2)). Consider the subgroup chains

$$(3.16) \quad G[p] \geq (pG)[p] \geq (p^2G)[p] \geq \dots \geq (p^nG)[p] = 0,$$

$$(3.17) \quad G = G[p^n] \geq G[p^{n-1}] \geq \dots \geq G[p] \geq 0.$$

Philip Hall has proved that a subgroup Γ of AG (G any group) is nilpotent if Γ stabilizes a finite subgroup chain of G , i.e. Γ induces the identity mapping in all factors of this chain [4, p. 787]. Since N_1 is a normal p -subgroup of AG , N_1 induces in $G[p]$ a group θ of automorphisms which stabilizes (3.16) (see (2.2)). Clearly $\theta \cong N_1/M$, where M is the set of all $\gamma \in N_1$ such that $\gamma|G[p] = 1$. One verifies that M stabilizes (3.17). Applying P. Hall's result it follows that $\theta \cong N_1/M$ and M both are nilpotent. Hence, N_1 is a solvable p -group and consequently (cf. [9, p. 190]),

$$(3.18) \quad N_1 \text{ is locally finite.}$$

It follows from (3.14) and (3.15) that every abelian subgroup of N_1 is cyclic. A finite p -group, $p \neq 2$, is cyclic if all its abelian normal subgroups are cyclic (cf. [8, p. 304]). Consequently, every finite subgroup of N_1 is cyclic, and because of (3.18), so is every finitely generated subgroup of N_1 . Hence N_1 is abelian and (3.14) and (3.15) imply that N_1 is cyclic. From Theorem B we obtain that N_1 is contained in the center of AG violating (3.12). Hence, the assumption (3.14) has been contradicted and P is not cyclic. Since the maximal p -subgroup of the center of AG is cyclic (cf. (2.1)) it follows that $P \not\leq ZAG$ and, because of (3.13), P is a noncentral normal subgroup of AG of exponent p which is contained in N . This completes the proof.

Proof of Theorem A. The automorphism group of a reduced abelian p -group G is commutative if and only if G is cyclic (cf. (2.1)). Since this, by hypothesis, is not the case, AG is a noncentral normal subgroup of itself. Therefore, (iv) implies (iii) which in turn implies (ii). Since every normal p -subgroup of AG fixes $G[p]/(pG)[p]$ elementwise (cf. (2.2)), (i) is a consequence of (ii). Theorem 3.2 completes the proof.

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